Birkhoff's theorem and scalar-tensor theories of gravitation

1977 J. Phys. A: Math. Gen. 10 993

(http://iopscience.iop.org/0305-4470/10/6/018)

J. Phys. A: Math. Gen., Vol. 10, No. 6, 1977. Printed in Great Britain. © 1977

Birkhoff's theorem and scalar–tensor theories of gravitation

K D Krori† and D Nandy‡

† Mathematical Physics Forum, Cotton College, Gauhati-1, Assam, India
‡ St Anthony's College, Shillong-3, India

Received 14 June 1976, in final form 3 February 1977

Abstract. We show here that a sufficient condition for Birkhoff's theorem to hold for the Sen–Dunn and Ross scalar–tensor theories of gravitation is that the scalar field should be time-invariant. An earlier result due to Reddy in the case of the Sen–Dunn theory is thus corrected.

1. Introduction

Schücking (1957) has shown that Birkhoff's theorem is valid in Jordan's (1952) extended theory of gravitation when the gravitational invariant of the theory is independent of time. Reddy (1973) obtained a similar result for the Brans–Dicke theory of gravitation. In the case of the Sen–Dunn theory, however, he has shown that Birkhoff's theorem is valid whatever may be the nature of the scalar field. Unfortunately Reddy's arguments are based on an incorrect set of equations for the Sen–Dunn theory. Here we find, from the corrected equations, the weaker result that Birkhoff's theorem holds if the scalar field of the Sen–Dunn theory is time-independent. The same holds also for the Ross (1972) scalar–tensor theory of gravitation.

We discuss the Sen–Dunn theory in § 2 and the Ross theory in § 3.

2. Birkhoff's theorem in Sen–Dunn theory

Birkhoff (1927) has shown that every spherically symmetric solution of the Einstein vacuum field equations is static. This fact is known as Birkhoff's theorem. We show here that this theorem holds in the case of Sen–Dunn theory when the scalar field is time-independent.

The field equations of Sen and Dunn (1971) for the combined scalar and tensor fields are

$$R_{ij}-\tfrac{1}{2}g_{ij}R=-8\pi G(x^0)^{-2}T_{ij}+\omega(x^0)^{-2}(x^0_{|i}x^0_{|j}-\tfrac{1}{2}g_{ij}x^0_{|k}x^{0|k}) \quad (1)$$

where $\omega=\frac{3}{2}$, T_{ij} is the energy-momentum tensor of the field and R is the usual Riemann curvature scalar. The single short vertical bar denotes partial differentiation.

We consider the spherically symmetric metric in the form

$$\frac{ds^2}{x^{02}}=-e^{\lambda}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)+e^{\nu}\,dt^2 \quad (2)$$

where

$$\lambda = \lambda(r, t); \qquad \nu = \nu(r, t) \tag{3}$$

and the scalar field

$$x^0 = x^0(r, t). \tag{4}$$

The field equations of Sen and Dunn in vacuum for the metric (2) are found as follows:

$$-\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{e^\lambda}{r^2}-3\frac{(x^0)'^2}{(x^0)^2}-\frac{4}{r}\frac{(x^0)'}{x^0}-\nu'\frac{(x^0)'}{x^0}+e^{\lambda-\nu}\left(2\frac{\ddot{x}^0}{x^0}-\frac{(\dot{x}^0)^2}{(x^0)^2}-\dot{\nu}\frac{\dot{x}^0}{x^0}\right)$$
$$=\frac{\omega}{2(x^0)^2}[e^{-\lambda}(x^0)'^2+e^{-\nu}(\dot{x}^0)^2] \tag{5}$$

$$-e^{-\lambda}\left[\frac{\nu''}{2}+\frac{1}{4}\nu'^2-\frac{1}{4}\lambda'\nu'+\frac{\nu'-\lambda'}{2r}+\frac{(x^0)'}{x^0}\left(\frac{2}{r}-(\lambda'-\nu')\right)+2\frac{(x^0)''}{x^0}-\frac{(x^0)'^2}{(x^0)^2}\right]$$
$$+e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{1}{4}\dot{\lambda}^2-\frac{1}{4}\dot{\lambda}\dot{\nu}+2\frac{\ddot{x}^0}{x^0}-\frac{(\dot{x}^0)^2}{(x^0)^2}+\frac{\dot{x}^0}{x^0}(\dot{\lambda}-\dot{\nu})\right)$$
$$=-\frac{\omega}{2(x^0)^2}[e^{-\lambda}(x^0)'^2-e^{-\nu}(\dot{x}^0)^2] \tag{6}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}-2\frac{(x^0)''}{x^0}+\frac{(x^0)'^2}{(x^0)^2}+\lambda'\frac{(x^0)'}{x^0}-\frac{4}{r}\frac{(x^0)'}{x^0}\right)+e^{-\nu}\left(\dot{\lambda}\frac{\dot{x}^0}{x^0}+3\frac{(\dot{x}^0)^2}{(x^0)^2}\right)+\frac{1}{r^2}$$
$$=-\frac{\omega}{2(x^0)^2}[e^{-\nu}(\dot{x}^0)^2+e^{-\lambda}(x^0)'^2] \tag{7}$$

$$-\frac{\dot{\lambda}}{r}-\dot{\lambda}\frac{(x^0)'}{x^0}+2\frac{(\dot{x}^0)'}{x^0}-\nu'\frac{\dot{x}^0}{x^0}-4\frac{(x^0)'\dot{x}^0}{x^{02}}=\frac{\omega}{(x^0)^2}(x^0)'\dot{x}^0 \tag{8}$$

where primes denote partial differentiation with respect to r and dots denote partial differentiation with respect to t.

In this case Birkhoff's theorem will immediately follow when x^0 is a constant. When however, the scalar field is a function of r alone, that is, $\dot{x}^0 = 0$, we have from (8) that either

$$\dot{\lambda} = 0 \qquad \text{or} \qquad x^0 = \frac{\alpha}{r}, \qquad \alpha = \text{constant}. \tag{9}$$

When $\dot{\lambda} = 0$, we have from (5),

$$-\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{e^\lambda}{r^2}-3\frac{(x^0)'^2}{(x^0)^2}-\frac{4}{r}\frac{(x^0)'}{x^0}-\nu'\frac{(x^0)'}{x^0}=\frac{\omega}{2(x^0)^2}e^{-\lambda}(x^0)'^2. \tag{10}$$

Partial differentiation of (10) with respect to t gives

$$\dot{\nu}'\left(\frac{(x^0)'}{x^0}+\frac{1}{r}\right)=0.$$

But since (9) is not the condition in this case $\dot{\nu}' = 0$. From this we may conclude that,

$$\nu = \psi(r)+\rho(t). \tag{11}$$

Now we can find a co-ordinate system in which $\nu \to \nu_T$ such that $\nu_T = \psi(r)$, while the other components of the metric tensor remain unchanged. For example, $x^i \to (x^i)'$ and $t \to f(t')$ is such a transformation. Hence Birkhoff's theorem will hold in this system.

But when $x^0 = \alpha/r$, we find from (5) and (7) that

$$e^{\lambda} = -1 \tag{12}$$

which gives an unphysical result. Thus we find that the time-invariance of the scalar field is a sufficient condition for Birkhoff's theorem to hold in the Sen–Dunn theory. This corrects an earlier result due to Reddy which asserts that Birkhoff's theorem holds irrespective of the nature of the scalar field in Sen–Dunn theory.

3. Birkhoff's theorem in Ross theory of gravitation

The Ross field equations for the regions of space which have no charge and mass densitites are

$$S_{\pi\beta} - \tfrac{1}{2} g_{\pi\beta} S = 0 \tag{13}$$

and

$$\Phi^{|\alpha}_{\|\alpha} = g^{\alpha\beta}\Phi_{|\alpha|\beta} - g^{\alpha\beta}\Phi_{|\pi}\left\{ {\pi \atop \alpha\beta} \right\} - 4\Phi_{|\alpha}\Phi^{|\alpha} = 0 \tag{14}$$

where

$$S_{\pi\beta} = R_{\pi\beta} - 2\Phi_{|\pi|\beta} - 2\Phi_{|\pi}\Phi_{|\beta} + 2g_{\pi\beta}\Phi^{|\alpha}\Phi_{|\alpha} - g_{\pi\beta}g^{\gamma\alpha}\Phi_{|\alpha|\gamma} + 2\Phi_{|\alpha}\left\{ {\alpha \atop \pi\beta} \right\} + g_{\pi\beta}g^{\gamma\alpha}\Phi_{|\delta}\left\{ {\delta \atop \alpha\gamma} \right\} \tag{15}$$

and

$$\Phi^{|\alpha} \equiv g^{\alpha\beta}\Phi_{|\beta}. \tag{16}$$

A double vertical bar here denotes covariant differentiation. Φ is the fundamental scalar field in the theory. $R_{\pi\beta}$ is the usual contracted Riemann curvature tensor and $\left\{ {\delta \atop \alpha\gamma} \right\}$ is the Christoffel symbol of the second kind.

We consider the spherically symmetric metric in the form

$$ds^2 = -e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}\, dt^2 \tag{17}$$

where

$$\lambda = \lambda(r, t); \qquad \nu = \nu(r, t) \tag{18}$$

and the scalar field

$$\Phi = \Phi(r, t). \tag{19}$$

The Ross vacuum field equations for the metric (17) can be written as follows:

$$-e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} + e^{-\lambda}\left(\frac{4\Phi'}{r} - 3\Phi'^2 + \Phi'\nu'\right) + e^{-\nu}(\dot{\Phi}^2 - 2\ddot{\Phi} + \dot{\Phi}\dot{\nu}) = 0 \tag{20}$$

$$-e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right) + e^{-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) + e^{\lambda}\left(2\Phi'' - \Phi'^2 + \frac{2\Phi'}{r} - \lambda'\Phi' + \Phi'\nu'\right)$$
$$+ e^{-\nu}(\dot{\Phi}^2 - 2\ddot{\Phi} + \dot{\Phi}\dot{\nu} - \dot{\Phi}\dot{\lambda}) = 0 \tag{21}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}+e^{-\lambda}\left(2\Phi''-\Phi'^2-\lambda'\Phi'+\frac{4\Phi'}{r}\right)-e^{-\nu}(\dot{\Phi}\dot{\lambda}-3\dot{\Phi}^2)=0 \tag{22}$$

$$-\frac{\dot{\lambda}}{r}-2\dot{\Phi}'-2\Phi'\dot{\Phi}+\Phi'\dot{\lambda}+\dot{\Phi}\nu'=0 \tag{23}$$

$$\ddot{\Phi}-\tfrac{1}{2}(\dot{\nu}-\dot{\lambda})\dot{\Phi}-8\dot{\Phi}^2=e^{\nu-\lambda}\left[\Phi''+\Phi'\left(\frac{2}{r}+\frac{\nu'-\lambda'}{2}\right)-8\Phi'^2\right]. \tag{24}$$

When Φ is a constant these field equations reduce to Einstein field equations in empty space and Birkhoff's theorem immediately follows.

When, however, the scalar field is a function of r alone, that is, $\dot{\Phi}=0$ we have from (23) either

$$\dot{\lambda}=0 \qquad \text{or} \qquad \Phi=\ln r+k, \qquad k=\text{constant}. \tag{25}$$

When $\lambda = 0$ we have from (20) and (22)

$$\Phi'(\lambda'+\nu')-\frac{\lambda'+\nu'}{r}-2(\Phi''+\Phi'^2)=0. \tag{26}$$

Partial differentiation of (26) with respect to t gives

$$\dot{\nu}'\left(\Phi'-\frac{1}{r}\right)=0.$$

As (25) does not hold in this case, $\dot{\nu}'=0$ which leads to $\nu=\eta(r)+\beta(t)$. As before we can find a coordinate system in which $\nu \to \nu_T$ such that $\nu_T=\eta(r)$ keeping other components of the metric tensor unchanged and hence Birkhoff's theorem will be valid.

Again when $\Phi=\ln r+k$ we find from (20) that $e^{\lambda}=0$ which gives an unphysical solution. This proves Birkhoff's theorem in the Ross theory of gravitation when the scalar field is time-invariant.

Considering the results obtained so far in the different scalar–tensor theories, one may suspect that the scalar field should be time-independent for Birkhoff's theorem to be valid for all of them.

Acknowledgment

The authors are grateful to the Government of Assam for all facilities provided at Cotton College, Gauhati-1, for carrying out this piece of work. One of them (DN) expresses his thanks to Mr A K Aditya for encouragement.

References

Birkhoff G D 1927 *Relativity and Modern Physics* (Cambridge, Mass.: Harvard University Press)
Jordan P 1952 *Schwerkraft und Weltall* (Braunschweig: Friedrich Vieweg und Sohn)
Reddy D R K 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 1867
Ross D K 1972 *Phys. Rev.* D **5** 284
Schücking E 1957 *Z. Phys.* **148** 72
Sen D K and Dunn K A 1971 *J. Math. Phys.* **12** 578